Binary Markov Planes

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Abstract

Binary Markov Plane is a two-dimensional construct with a property of continuous Markov process along any line. While one-dimensional case can be analysed easily, a two-dimensional case has conceptually difficult problems. Possibly a lack of proper mathematical apparatus (or author's unawareness of it) leaves some problems open. Numerical simulations are carried out to support the analytical results. A practical application of the results is discussed.

Contents

1. Notations and Rules

Consider a binary random function *f* defined on some space. Let function values be either zero or one. We may call them white for zero and black for one. If some point is selected on the space, the value of the function at this point is either zero or one, or in other words the selected point is either white or black. Since the function is random there is a probability associated with the point. This probability defines the information we have about the point. Let probability be the chances of the point being black, so its value is between zero and one, with the meaning that if it is zero, then we know that the point is white, and if its is one, then we know that the point is black. We will call this value the state of the point.

Let us denote points with small bold letters like **x**; distances and other values with italic letters like *x*; conditional probability with square brackets; and negation with a bar above a letter. The point letter used inside the square brackets specifies the black point. And a point letter with a *bar* used inside the square brackets specifies the white point. For example:

$$[\bar{\mathbf{x}} \mid \mathbf{y}] = a$$

reads: probability to find point **x** white knowing that point **y** is black equals to *a*.

Rules of inference such as:

$$[\mathbf{xy} \mid \mathbf{z}] = [\mathbf{x} \mid \mathbf{yz}][\mathbf{y} \mid \mathbf{z}]$$
$$[\mathbf{x} \mid \mathbf{z}] = [\mathbf{xy} \mid \mathbf{z}] + [\mathbf{x}\bar{\mathbf{y}} \mid \mathbf{z}]$$
$$[\mathbf{x} \mid \mathbf{y}] + [\bar{\mathbf{x}} \mid \mathbf{y}] = 1$$

are used extensively.

2. Line

Definition D1. Let random function *f* defined on a line have a property that finding two infinitesimally close points of different states (black and white) depends only on and is proportional to the distance between the points.

Lemma L1. D1 defines a construction with Markovian property – the knowledge of the point states matters only for the closest points on each side.

Let us chose a sequence of points **a**,**b**,**c**,**d**,**e** on the line so that **a** is on the left of **b**, **b** is on the left of **c**, and so on. L1 above states that if we know states of points **b** and **d**, the probability value of **c** does not depend on knowledge about **a** or **e**.

Let the state of a point without any knowledge be equal to *U*. This value equals to the probability to find a random point **x** black: $[\mathbf{x}] = U$. Also this value equals to the ratio between total black area and total area:

$$Z^{-1} \int f(x)\, dx = U$$

Here *Z* is a normalization: $Z^{-1} \int dx = 1$. The probability to find a random point **x** white is $[\bar{\mathbf{x}}] = 1 - U = T$.

Let us define the coefficient of proportionality from D1 as *nUT*, where *n* is some value. This form is selected because finding a pair **x** and **y** of different states is proportional to both *U* and *T*:

$$[\bar{\mathbf{x}}\mathbf{y}] = [\bar{\mathbf{x}} \mid \mathbf{y}][\mathbf{y}] = [\bar{\mathbf{x}} \mid \mathbf{y}]U$$
$$[\bar{\mathbf{x}}\mathbf{y}] = [\mathbf{y} \mid \bar{\mathbf{x}}][\bar{\mathbf{x}}] = [\mathbf{y} \mid \bar{\mathbf{x}}]T$$

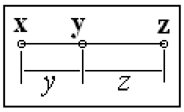
So for a small distance Δ:

$$[\overline{\Delta} \mid \mathbf{x}] = nT\Delta$$
$$[\Delta \mid \overline{\mathbf{x}}] = nU\Delta$$

The meaning of value n, as will be shown later, is that, while U and T values give amount of black and white, $(nT)^{-1}$ and $(nU)^{-1}$ give the average size of black and white lags.

Problem P1. Find the state of the point **y** at a distance y from the point **x**, given **x** is black, ($[\mathbf{y} \mid \mathbf{x}] = ?$)
Answer: $[\mathbf{y} \mid \mathbf{x}] = U + Te^{-ny} = A(y)$

Problem P2. Find the state of the point **y** between black points **x** and **z** distanced from **y** by y and z correspondingly, ($[\mathbf{y} \mid \mathbf{xz}] = ?$).



Answer: $[\mathbf{y} \mid \mathbf{xz}] = \dfrac{A(y)A(z)}{A(y+z)}$

Problem P3. Find the probability to find black line of size y between points **x** and **y**, given **x** is black.
Answer: $[\text{line}(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}] = e^{-nTy}$

Problem P1

Consider three points **x**, **y**, and **Δ**, with the following properties. The distance between **x** and **y** is y; the distance between **y** and **Δ** is Δ; **y** lays between **x** and **Δ**; and Δ is infinitesimally small. Let $A(\mathbf{xy}) = A(y) = A_y = [\mathbf{y} \mid \mathbf{x}]$. Then

$$A(y + \Delta) = [\mathbf{\Delta} \mid \mathbf{x}] = [\mathbf{\Delta y} \mid \mathbf{x}] + [\mathbf{\Delta \overline{y}} \mid \mathbf{x}] = [\mathbf{\Delta} \mid \mathbf{xy}][\mathbf{y} \mid \mathbf{x}] + [\mathbf{\Delta} \mid \mathbf{x\overline{y}}][\overline{\mathbf{y}} \mid \mathbf{x}]$$

By definition the probability state of **Δ** depends only on the state of **y** and distance Δ:

$$A(y + \Delta) = [\mathbf{\Delta} \mid \mathbf{xy}][\mathbf{y} \mid \mathbf{x}] + [\mathbf{\Delta} \mid \mathbf{x\overline{y}}][\overline{\mathbf{y}} \mid \mathbf{x}] = [\mathbf{\Delta} \mid \mathbf{y}][\mathbf{y} \mid \mathbf{x}] + [\mathbf{\Delta} \mid \overline{\mathbf{y}}](1 - [\mathbf{y} \mid \mathbf{x}]) =$$
$$= (1 - nT\Delta)A_y + nU\Delta(1 - A_y) = A_y + \Delta(nU - nA_y)$$

Expanding left-hand side into a Taylor series we find the differential equation:

$$A'_y = nU - nA_y$$

Given the initial condition $A(0) = 1$ the equation above solves into:

$$A_y = U + Te^{-ny}$$

which gives the answer to P1.

Problem P1 – autocorrelation

Now let us obtain the same result assuming that the autocorrelation function has exponential form:

$$Z^{-1}\int \tilde{f}\ \tilde{f}_y\, dx = Be^{-ny}$$

where *B* and *n* are some values; and $\tilde{f}$ is a normalized by $Z^{-1}\int \tilde{f}\, dx = 0$ function so that $\tilde{f} = f - U$. For short here we denote $f = f(x)$ and $f_y = f(x+y)$. Expectation of two points **x** and **y** with a distance *y* between them can be written as an integral over the space:

$$[\mathbf{xy}] = Z^{-1}\int f\ f_y\, dx = Z^{-1}\int \left(U + \tilde{f}\right)\left(U + \tilde{f}_y\right)dx =$$
$$= U^2 + Z^{-1}\int \tilde{f}\ \tilde{f}_y\, dx = U^2 + Be^{-ny}$$

If *y* is set to zero, then

$$U^2 + B = Z^{-1}\int f\ f_{y=0}\, dx = Z^{-1}\int f^2\ dx = Z^{-1}\int f\ dx = U$$

From where *B* is found to be *UT*. So the final result is:

$$[\mathbf{y}\,|\,\mathbf{x}] = \frac{[\mathbf{xy}]}{[\mathbf{x}]} = \frac{1}{U}\left(U^2 + UTe^{-ny}\right) = U + Te^{-ny}$$

The same conditional probabilities are obtained using the exponential form of autocorrelation function.

Transitivity

If a point **x** is randomly selected its expected state is *U*. This means that we have no knowledge for a point with a state *U*. When the point's state is different than *U*, say *p*, we have some knowledge about points neighbourhood. So the state *q* of a point **y** distanced from **x** by *y* is:

$$q = [\mathbf{y}\,|\,\mathbf{x}]p + [\mathbf{y}\,|\,\bar{\mathbf{x}}](1 - p)$$

The first term in square brackets is *A*(*y*), P1. The second term is easily found by:

$$[\mathbf{y}\,|\,\bar{\mathbf{x}}] = \frac{[\mathbf{y}\bar{\mathbf{x}}]}{[\bar{\mathbf{x}}]} = \frac{[\bar{\mathbf{x}}\,|\,\mathbf{y}][\mathbf{y}]}{[\bar{\mathbf{x}}]} = \frac{[\mathbf{y}]}{[\bar{\mathbf{x}}]}\left(1-[\mathbf{x}\,|\,\mathbf{y}]\right) = \frac{U}{T}\left(1-U-Te^{-ny}\right) = U-Ue^{-ny}$$

Note, that we used a symmetry identity $[\mathbf{y}\,|\,\mathbf{x}]=[\mathbf{x}\,|\,\mathbf{y}]$. If a state of a point is presented as a vector of two values $(p,1\text{-}p)^{\mathrm{T}}$, where the first number is the probability of black and the second is the probability of white, then it is possible to write a conversion in a matrix form:

$$\mathbf{M}(y) = \begin{pmatrix} [\mathbf{y}\,|\,\mathbf{x}] & [\mathbf{y}\,|\,\bar{\mathbf{x}}] \\ [\bar{\mathbf{y}}\,|\,\mathbf{x}] & [\bar{\mathbf{y}}\,|\,\bar{\mathbf{x}}] \end{pmatrix} = \begin{pmatrix} U+Te^{-ny} & U-Ue^{-ny} \\ T-Te^{-ny} & T+Ue^{-ny} \end{pmatrix}$$

and

$$(q,1-q)^T = \mathbf{M}(y)(p,1-p)^T$$

It is easy to see that the state $(U,T)^{\mathrm{T}}$ is an eigenvector of the above matrix. Another important observation is that for any point **z** further down from **y** by *z* (say, **y** is on the right size of **x** and **z** is on the right side of **y**) the knowledge about the point **x** (from the distance y+z) in the sate *p* is exactly the same as the knowledge about the point **y** in the state *q*. Or in other words, knowing **x** in state *p* makes **y** in state *q* be in unknown state. If **y**'s state changed to $q' \neq q$ (or fixed in *q* regardless of **x** in *p*), then new information is obtained in relation to the point in question **z.** The knowledge of **x** in state *p* becomes irrelevant in Markovian case. The condition for the equivalence of knowledge about **x** and **y** is transitivity of **M**: $\mathbf{M}(y+z)=\mathbf{M}(y)\mathbf{M}(z)$ which can be verified by direct matrix multiplication:

$$\begin{pmatrix} U+Te^{-ny} & U-Ue^{-ny} \\ T-Te^{-ny} & T+Ue^{-ny} \end{pmatrix}\begin{pmatrix} U+Te^{-nz} & U-Ue^{-nz} \\ T-Te^{-nz} & T+Ue^{-nz} \end{pmatrix} = \begin{pmatrix} U+Te^{-n(y+z)} & U-Ue^{-n(y+z)} \\ T-Te^{-n(y+z)} & T+Ue^{-n(y+z)} \end{pmatrix}$$

Transitivity and Markovinaity

Proving transitivity does not prove Markov property. It just establishes equality:

$$[\mathbf{x}\,|\,\mathbf{y}][\mathbf{y}\,|\,\mathbf{z}]+[\mathbf{x}\,|\,\bar{\mathbf{y}}][\bar{\mathbf{y}}\,|\,\mathbf{z}] = [\mathbf{x}\,|\,\mathbf{yz}][\mathbf{y}\,|\,\mathbf{z}]+[\mathbf{x}\,|\,\bar{\mathbf{y}}\mathbf{z}][\bar{\mathbf{y}}\,|\,\mathbf{z}]$$

from which relations $[\mathbf{x}\,|\,\mathbf{y}]=[\mathbf{x}\,|\,\mathbf{yz}]$ and $[\mathbf{x}\,|\,\bar{\mathbf{y}}]=[\mathbf{x}\,|\,\bar{\mathbf{y}}\mathbf{z}]$ do not follow.

If one assumes $[\mathbf{x}\,|\,\mathbf{yz}]=A_y\,b(y,z)$, where *b* is some symmetric function with the condition being equal to 1 on any zero and infinity borders, then both the solution P1 and transitivity condition hold, but Markovian property does not.

In the right hand side of the equation above *b* cancels out:

$$[\mathbf{x} \mid \mathbf{yz}] = A_y b$$
$$[\mathbf{y} \mid \mathbf{z}] = A_z$$
$$[\mathbf{x} \mid \bar{\mathbf{y}}\mathbf{z}] = \frac{A_{y+z}}{1 - A_z}\left(1 - \frac{A_z}{A_{y+z}} A_y b\right)$$
$$[\bar{\mathbf{y}} \mid \mathbf{z}] = 1 - A_z$$

At this point one can see that conditional probability $A(y)$, P1, is calculated and transitivity property is established, but L1 (Markovianity) is still an open question. L1 can be proven by following the same line of arguments as above for the solution P1, but instead finding $[\mathbf{z} \mid \mathbf{xy}]$ and $[\mathbf{z} \mid \bar{\mathbf{x}}\mathbf{y}]$. The key point is using the statement from the definition that the probability for the infinitesimally close points depends *only* on the distance between them. So the knowledge about the point **x** does not change the differential equation. And the result acquired is:

$$[\mathbf{z} \mid \mathbf{xy}] = [\mathbf{z} \mid \bar{\mathbf{x}}\mathbf{y}] = A(z) = [\mathbf{z} \mid \mathbf{y}]$$

which proves L1.

Problem P2

Now using proven Markovianity finding solution P2 is straightforward:

$$[\mathbf{y} \mid \mathbf{xz}] = \frac{[\mathbf{xy} \mid \mathbf{z}]}{[\mathbf{x} \mid \mathbf{z}]} = \frac{[\mathbf{x} \mid \mathbf{yz}][\mathbf{y} \mid \mathbf{z}]}{[\mathbf{x} \mid \mathbf{z}]} = \frac{A(y)A(z)}{A(y+z)}$$

Problem P3

Let $g(y)$ be the probability to find a black segment of size y between point **x** and **y** given **x** is black. A small increment near point **y** gives:

$$g(y + \Delta) = g(y)[\Delta \mid \mathbf{y}] = g(y)(1 - nT\Delta)$$

which gives the differential equation:

$$g'(y) = -nTg(y)$$

Knowing that $g(0)=1$ the solution for g is

$$g(y) = e^{-nTy}$$

which is the solution P3.

The same solution can be found by considering a segment consisting of a sequence of points $\mathbf{y}_i$. Let us break the segment into N sections. Each i-th section ends with a point y_i. Then the function g can be written as:

$$g(y) = \lim_{N\to\infty}[\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3 \ldots \mathbf{y}_{N-1}\mathbf{y}_N \mid \mathbf{x}] = \lim_{N\to\infty}[\mathbf{y}_1 \mid \mathbf{x}][\mathbf{y}_2 \mid \mathbf{y}_1]\ldots[\mathbf{y}_N \mid \mathbf{y}_{N-1}] =$$

$$= \lim_{N\to\infty}[\mathbf{y}_1 \mid \mathbf{x}]^N = \lim_{N\to\infty}\left(1 - nT\frac{y}{N}\right)^N = e^{-nTy}$$

which is the same as the solution above.

Two points

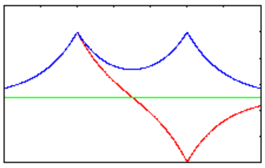


Figure 1 Probability distribution given two points

The figure on the right shows the distribution of state given two points. Two peaks of blue line correspond to the case when two points are black. The value there is 1. The green line is the U level. The values between points correspond to the solution of the P2. The values outside is the solution of the P1, which asymptotically reduce to the value U. The red line shows the solution when the right point $\mathbf{y}$ is white. On the left from the point $\mathbf{x}$ the solution is the same as the blue line. And in the middle the solution is found as

$$[\mathbf{y} \mid \mathbf{x}\overline{\mathbf{z}}] = \frac{[\mathbf{y}\overline{\mathbf{z}} \mid \mathbf{x}]}{[\overline{\mathbf{z}} \mid \mathbf{x}]} = \frac{[\overline{\mathbf{z}} \mid \mathbf{y}][\mathbf{y} \mid \mathbf{x}]}{[\overline{\mathbf{z}} \mid \mathbf{x}]} = \frac{(1 - A_z)A_y}{1 - A_{y+z}}$$

3. Plane

Definition D2. Markov plane is a two dimensional construction for which any selected line reveals the property of definition for one dimensional case.

The state surface around a black point must be circular symmetric because no direction is more preferable. The solution along any line passing through the point must correspond to one-dimensional case, P1.

An interesting question is how the state surface looks given two black points. Imagine fixed points $\mathbf{x}$ and $\mathbf{z}$, and the moving point $\mathbf{y}$. When $\mathbf{y}$ crosses the line between $\mathbf{x}$ and $\mathbf{z}$, it must have the solution of P2. When **y** crosses the line going through **x** and **z**, but outside of the segment between them, then it must have the solution of P1. In one-dimensional case we distinguish these two cases by knowing the order of the points on the line. In two-dimensional case that is not easy, because there is no order for arbitrary chosen points. Having solution of P1 means that one of the points hides behind the other and its state does not influence the state of the point in question.

Triangle Theorem. State of a point **x** given two black points **y** and **z** can be calculated as:

$$[\mathbf{x}\,|\,\mathbf{yz}] = \frac{A(x)A(y)}{A\left(\frac{1}{2}(x+y+z)\right)}$$

where y is distance between **x** and **y**, x distance between **x** and **z**, and z distance between **y** and **z**.

Consider a function

$$Q(\mathbf{xyz}) = Q(x,y,z) = \frac{[\mathbf{x}\,|\,\mathbf{y}][\mathbf{x}\,|\,\mathbf{z}]}{[\mathbf{x}\,|\,\mathbf{yz}]}$$

It is easy to prove that this function is symmetric for all permutations of its arguments. For example:

$$Q(x,y,z) = \frac{[\mathbf{x}\,|\,\mathbf{y}][\mathbf{x}\,|\,\mathbf{z}]}{[\mathbf{x}\,|\,\mathbf{yz}]} = \frac{[\mathbf{x}\,|\,\mathbf{y}][\mathbf{x}\,|\,\mathbf{z}][\mathbf{y}\,|\,\mathbf{z}]}{[\mathbf{x}\,|\,\mathbf{yz}][\mathbf{y}\,|\,\mathbf{z}]} =$$

$$= \frac{[\mathbf{x}\,|\,\mathbf{y}][\mathbf{x}\,|\,\mathbf{z}][\mathbf{y}\,|\,\mathbf{z}]}{[\mathbf{y}\,|\,\mathbf{xz}][\mathbf{x}\,|\,\mathbf{z}]} = \frac{[\mathbf{y}\,|\,\mathbf{x}][\mathbf{y}\,|\,\mathbf{z}]}{[\mathbf{y}\,|\,\mathbf{xz}]} = Q(y,x,z)$$

On the other hand

$$Q(x,y,x+y) = \frac{[\mathbf{y}\,|\,\mathbf{x}][\mathbf{y}\,|\,\mathbf{z}]}{[\mathbf{y}\,|\,\mathbf{xz}]} = \frac{A(y)A(x+y)}{[\mathbf{y}\,|\,\mathbf{x}]} = A(x+y)$$

because when z=x+y, points **x**, **y**, and **z** are collinear and $[\mathbf{y}\,|\,\mathbf{xz}] = [\mathbf{y}\,|\,\mathbf{x}]$ as **x** is between **y** and **z**. The relation above implies that the first two arguments may be modified:

$$Q(x,y,x+y) = A(x+y) = Q(x-a, y+a, x+y)$$

where a is an arbitrary value. Since a is arbitrary, Q must depend only on the sum of the first and the second arguments: $Q(x,y,z) = Q(x+y,z)$. And since Q is symmetric the same is valid for any pair of its arguments, so it must depend on the sum of all its arguments: $Q(x,y,z) = Q(x+y+z)$.

The only acceptable solution for Q is then

$$Q(\mathbf{xyz}) = Q(x, y, z) = A\left(\frac{1}{2}(x + y + z)\right)$$

because

$$Q(x, y, x + y) = A\left(\frac{1}{2}(x + y + x + y)\right) = A(x + y)$$

Inserting the solution for Q into the definition of Q gives the desired result of the theorem:

$$[\mathbf{x} \mid \mathbf{yz}] = \frac{A(\mathbf{xy})A(\mathbf{xz})}{Q(\mathbf{xyz})}$$

Figure 3 shows conditional probability for two black points.

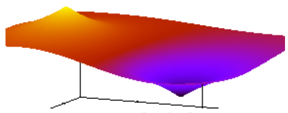


Figure 5 Probability distribution given black and white points

Problem P4. Find $[\mathbf{x} \mid \mathbf{y}\bar{\mathbf{z}}]$.

Answer: $[\mathbf{x} \mid \mathbf{y}\bar{\mathbf{z}}] = [\mathbf{x} \mid \mathbf{y}]\frac{1 - [\mathbf{z} \mid \mathbf{xy}]}{1 - [\mathbf{z} \mid \mathbf{y}]}$; note that

$[\mathbf{x} \mid \mathbf{yz}] = [\mathbf{x} \mid \mathbf{y}]\frac{[\mathbf{z} \mid \mathbf{xy}]}{[\mathbf{z} \mid \mathbf{y}]}$

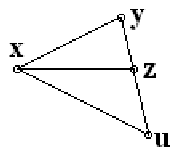


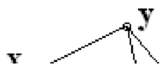
Figure 6 Problem P5

Problem P5. Find $[\mathbf{x} \mid \mathbf{yzu}]$, where **y,z,u** are collinear.

Answer: $[\mathbf{x} \mid \mathbf{yzu}] = \frac{A(\mathbf{xy})A(\mathbf{xz})A(\mathbf{xu})}{Q(\mathbf{xyz})Q(\mathbf{xzu})}$

Open Problem P6. Find $[\mathbf{x} \mid \mathbf{yzu}]$ for any **y,z,u**.

Possible Answer: $[\mathbf{x} \mid \mathbf{yzu}] = \frac{A(\mathbf{xy})A(\mathbf{xz})A(\mathbf{xu})}{Q(\mathbf{xyz})Q(\mathbf{xzu})} \cdot \frac{Q(\mathbf{yzu})}{A(\mathbf{yu})}$



Open Problem P7. Find $[\mathbf{x} \mid \{\mathbf{y}_i\}]$ and $[\{\mathbf{y}_i\} \mid \mathbf{x}]$, where $\{y_i\}$ is a set of black and white points.

Open Problem P8. Find $[\mathbf{x} \mid \mathit{curve}(\mathbf{a})]$ and $[\mathit{curve}(\mathbf{a}) \mid \mathbf{x}]$, where **x** may or may not belong to the curve **a**.

Open Problem P9. Find $[\mathit{loop}(\mathbf{b}) \mid \mathbf{x}]$, where **x** belongs to a hollow shape **b** – convex closed curve.

Open Problem P10. Find $[shape(\mathbf{c}) \,|\, \mathbf{x}]$, where **x** belongs to a solid convex shape **c**.

Conjecture C10: Solution to P10 is

$$[shape(\mathbf{c}) \,|\, \mathbf{x}] = \exp -\frac{1}{2} nT\left(Perimeter(\mathbf{c}) + nU \cdot Area(\mathbf{c})\right)$$